Available online at www.sciencedirect.com

SciVerse ScienceDirect

Procedia - Social and Behavioral Sciences 80 (2013) 608 – 632

Procedia
Social and Behavioral Sciences

The 20th International Symposium on Transportation and Traffic Theory (ISTTT 2013)

A Path-size Weibit Stochastic User Equilibrium Model

Songyot Kitthamkesorn, Anthony Chen*

Department of Civil and Environmental Engineering, Utah State University
Logan, UT 84322-4110, USA

Abstract

The aim of this paper is to develop a path-size weibit (PSW) route choice model with an equivalent mathematical programming (MP) formulation under the stochastic user equilibrium (SUE) principle that can account for both route overlapping and route-specific perception variance problems. Specifically, the Weibull distributed random error term handles the identically distributed assumption such that the perception variance with respect to different trip lengths can be distinguished, and a path-size factor term is introduced to resolve the route overlapping issue by adjusting the choice probabilities for routes with strong couplings with other routes. A multiplicative Beckmann's transformation (MBec) combined with an entropy term are used to develop the MP formulation for the PSW-SUE model. A path-based algorithm based on the partial linearization method is adopted for solving the PSW-SUE model. Numerical examples are also provided to illustrate features of the PSW-SUE model and its differences compared to some existing SUE models as well as its applicability on a real-size network.

Selection and peer-review under responsibility of Delft University of Technology

Keywords: Stochastic user equilibrium; random utility model; Weibull distribution; logit model; path-size weibit model

1. Introduction

The stochastic user equilibrium (SUE) model is well-known in the literature. It relaxes the perfect information assumption of the deterministic user equilibrium model by incorporating a random error term in the route travel cost function to simulate travelers' imperfect perceptions of travel costs. Route choice models under this approach can have different specifications according to the modeling assumptions on the random error term. The two commonly used random error terms are Gumbel (Dial, 1971) and Normal (Daganzo and Sheffi, 1977) distributions, corresponding to the multinomial logit (MNL) and multinomial probit (MNP) route choice models, respectively. MNL model has a closed-form probability expression and can be formulated as an equivalent mathematical programming formulation (MP) by using an entropy-type model for the logit-based SUE problem (Fisk, 1980). The drawbacks of the MNL model are: (1) inability to account for overlapping (or correlation) among routes and (2) inability to account for perception variance with respect to trips of different lengths. These two drawbacks stem from the underlying assumptions that the random error terms are *independently and identically distributed* (IID) with the same and fixed perception variance (Sheffi, 1985). MNP route choice model, on the other hand, does not have such drawbacks, because it handles the route overlapping and identical perception variance problems between

* Corresponding author. Tel.: +1-435-797-7109; fax: +1-435-797-1185.
E-mail: anthony.chen@usu.edu (A. Chen).

1877-0428 © 2013 The Authors. Published by Elsevier Ltd. Open access under CC BY-NC-ND license.
Selection and peer-review under responsibility of Delft University of Technology
doi:10.1016/j.sbspro.2013.05.033

routes by allowing the covariance between random error terms for pairs of routes. However, the MNP model does not have a closed-form solution and it is computationally burdensome when the choice set contains more than a handful of routes. Due to the lack of a closed-form probability expression, solving the MNP model will require either Monte Carlo simulation (Sheffi and Powell, 1982), Clark's approximation method (Maher, 1992), or numerical method (Rosa and Maher, 2002).

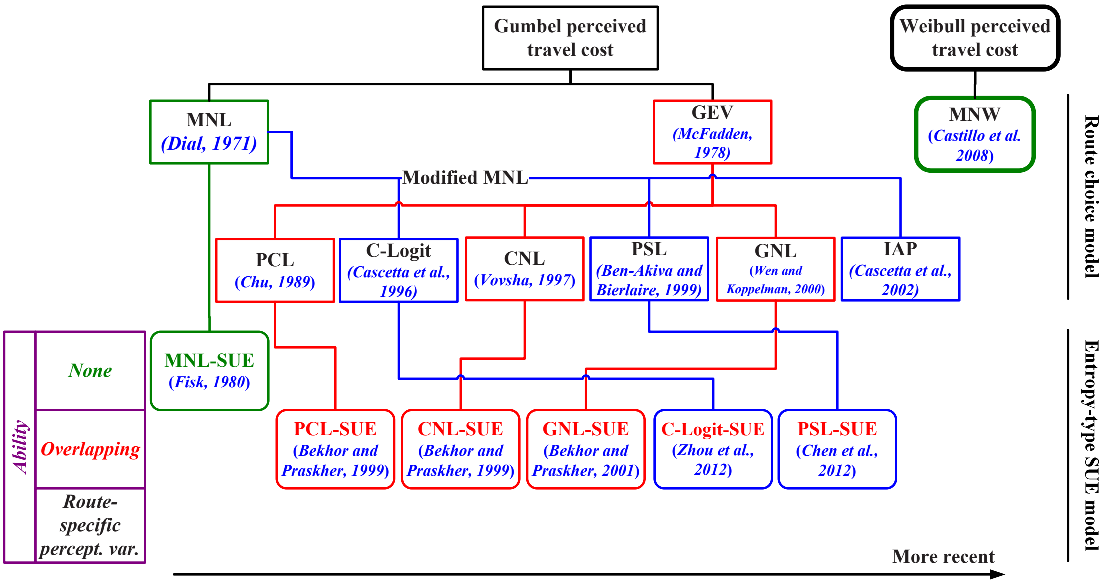


Fig. 1. Chronicle development of some closed-form route choice models and their MP formulations

To overcome the deficiencies of the MNL model, some analytical closed-form extensions have been proposed in the literature. These models can be broadly classified into two groups: extended logit models and Weibull-based model as shown in Fig. 1. The extended logit models were developed mainly to handle the route overlapping problem. These models modified either the deterministic term or the random error term in the additive disutility function of the MNL model while maintaining the Gumbel distributed random error term assumption. The models modifying the deterministic term of the disutility function include the C-logit model (Cascetta *et al*., 1996), the implicit availability/perception (*IAP*) model (Cascetta *et al*., 2002), and the path-size logit (*PSL*) model (Ben-Akiva and Bierlaire, 1999). All three models add a correction term to the deterministic term of the disutility function to adjust the choice probability; however, the interpretation of each model is different. The C-logit model uses the commonality factor to penalize the coupling routes, while both the *IAP* and *PSL* models use a logarithmic correction term to modify the disutility (hence, the choice probability). Equivalent MP formulation for the length-based C-logit model was recently provided by Zhou *et al*. (2012). The models modifying the random error term of the disutility function include the cross-nested logit (*CNL*) model (Bekhor and Prashker, 1999), the paired combinatorial logit (*PCL*) model (Bekhor and Prashker, 1999), and the generalized nested logit (*GNL*) model (Bekhor and Prashker, 2001). These models are based on the generalized extreme value (*GEV*) theory (McFadden, 1978) using a two-level tree structure to capture the similarity among routes through the random error component of the disutility function. Equivalent MP formulations for all three models were given by Bekhor and Prashker (1999; 2001). Recall that the extended logit models with closed-form solution discussed above were developed to mainly address the *independence* assumption (i.e., route overlapping problem) of the MNL-SUE model. The *identically distributed* assumption (i.e., homogeneous perception variance problem) is still inherited in these extended logit-based SUE models. In other words, the perception variance is fixed (or constant) with respect to trips of different lengths over all routes and all origin-destination (O-D) pairs. In view of network equilibrium assignment, the *identically distributed* assumption seems unrealistic since it does not distinguish trip lengths of different O-D pairs. Hence, Chen *et al.* (2012) suggested a practical approach to partially relax the assumption by scaling the perception variance of an individual O-D pair. The *individual O-D specific scaling factors* allow the perception variance to increase or decrease according to the travel distance of each O-D pair. Specifically, the systematic disutility in the logit-based SUE models can be scaled appropriately to reflect different O-D trip lengths in a network by replacing a single dispersion parameter for all O-D pairs with *individual O-D dispersion parameters* for each O-D pair. However, it should be noted that it is not possible to scale individual routes of the same O-D pair since it would violate the logit-based SUE models' assumption of an equal variance across the routes within the same O-D pair. For a more comprehensive review of the extended logit models used in the SUE problem, readers are directed to the reviews given by Prashker and Bekhor (2004) and Chen *et al.* (2012).

On the other hand, Castillo *et al*. (2008) proposed the multinomial weibit (*MNW*) model to address the *identically distributed* assumption. This model assumes that the perceived route travel time follows the Weibull distribution, instead of the conventional Gumbel distribution. Under the *independence* assumption, the MNW model has a simple analytical form with route-specific perception variance (i.e., non-identical perception variances with respect to trips of different lengths). However, no equivalent MP formulation has been provided for the MNW-SUE model.

To our best knowledge, no closed-form probability expression with an equivalent MP formulation has been provided to simultaneously address both route overlapping and route-specific perception variance problems in the literature. The purpose of this paper is to provide a path-size weibit (*PSW*) route choice model with an equivalent MP SUE formulation that can account for both route overlapping and route-specific perception variance problems. Specifically, the Weibull distributed random error term handles the *identically distributed assumption* such that the perception variance with respect to different trip lengths can be distinguished, and a path-size factor term is introduced to resolve the *route overlapping issue* by adjusting the choice probabilities for routes with strong couplings with other routes. A *multiplicative* Beckmann's transformation (*MBec*) combined with an entropy term are used to develop the MP formulation for the PSW-SUE model. Some qualitative properties of the *PSW-SUE* formulation are rigorously proved. A path-based algorithm based on the partial linearization method is adopted for solving the PSW-SUE model. Numerical examples are also provided to illustrate features of the PSW-SUE model and its differences compared to some existing SUE models as well as its applicability on a real-size network.

The remaining of this paper is organized as follows. Section 2 provides some background of the MNW route choice model and develops the PSW model. In section 3, equivalent MP formulations for the MNW-SUE and PSW-SUE models are provided along with some qualitative properties. Section 4 presents a path-based algorithm for solving the SUE formulations. Numerical results are presented in Section 5, and some concluding remarks are provided in Section 6.

2. Weibit route choice models

In this section, we provide some background of the multinomial weibit[1] (MNW) route choice model and the development of the path-size weibit (PSW) model. Specifically, we show how the MNW model resolves the identical perception variance issue inherited in the classical multinomial logit (MNL) model. Then, a path-size factor is introduced to the MNW random utility maximization (RUM) model to develop the PSW model that can address both route overlapping and non-identical route perception variance problems.

2.1 MNW model

Castillo *et al*. (2008) developed the multinomial weibit (MNW) model as a route choice model without the *identically distributed* assumption typically required for the random error term in a random utility model (RUM) (e.g., the IID Gumbel distribution in the MNL model). Unlike the MNL model which used the conventional *additive* RUM (ARUM), the MNW model adopts the *multiplicative* RUM (MRUM) with the Weibull distribution as the random error term (Fosgerau and Bierlaire, 2009). The MNW disutility function can be written as

[1] The term "weibit" stands for "*Weib*ull probability un*it*". Note that this term has also been used in other disciplines, for example, the bioassay (Lonney, 1983), contingent valuation model in economics (Genius and Strazzera, 2002), and reliability engineering (Strong *et al*., 2009).

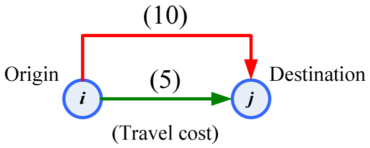
$$U_r^{ij} = \left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \tag{1}$$

where IJ is the set of O-D pairs, R_{ij} is the set of routes between O-D pair ij, g_r^{ij} is the (generalized) travel cost on route r between O-D pair ij, ε_r^{ij} is the *independently* Weibull distributed random error term on route r between O-D pair ij whose CDF is $F_{\varepsilon_r^{ij}}(t) = 1 - \exp(-t)$, $\zeta^{ij} \in \left[0, g_r^{ij}\right)$ is the location parameter, which identifies the lower bound of route perceived travel cost between O-D pair ij, and $\beta^{ij} \in (0, \infty)$ is the shape parameter, which is related to the route perception variance between O-D pair ij. Then, the MNW probability can be determined by

$$\begin{aligned} P_r^{ij} &= \Pr\left(\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \le \left(g_l^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_l^{ij}, \forall l \ne r\right), \forall r \in R_{ij}, ij \in IJ \\ &= \Pr\left(\frac{\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{\left(g_l^{ij} - \zeta^{ij}\right)^{\beta^{ij}}} \varepsilon_r^{ij} \le \varepsilon_l^{ij}, \forall l \ne r\right), \forall r \in R_{ij}, ij \in IJ, \end{aligned} \tag{2}$$

which gives the MNW route choice probability expression (see **Appendix A** for details):

$$P_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \tag{3}$$



MNL model

$(\theta = 0.5)$;

$$P_l^{ij} = \frac{e^{-2.5}}{e^{-2.5} + e^{-5}} = \frac{1}{1 + e^{-2.5}} = 0.924$$

MNW model

$(\beta^{ij} = 2.1, \zeta^{ij} = 0)$;

$$P_l^{ij} = \frac{5^{-2.1}}{5^{-2.1} + 10^{-2.1}} = \frac{1}{1 + (10/5)^{-2.1}} = 0.811$$

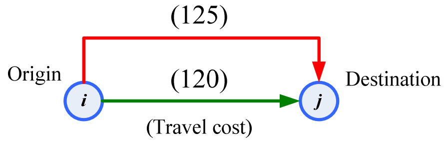
$(\beta^{ij} = 3.7, \zeta^{ij} = 0)$;

$$P_l^{ij} = \frac{5^{-3.7}}{5^{-3.7} + 10^{-3.7}} = \frac{1}{1 + (10/5)^{-3.7}} = 0.929$$

$(\beta^{ij} = 2.1, \zeta^{ij} = 2.5)$;

$$P_l^{ij} = \frac{2.5^{-2.1}}{2.5^{-2.1} + 7.5^{-2.1}} = \frac{1}{1 + (7.5/2.5)^{-2.1}} = 0.909$$

a) Short network



MNL model

$(\theta = 0.5)$;

$$P_l^{ij} = \frac{e^{-60}}{e^{-60} + e^{-62.5}} = \frac{1}{1 + e^{-2.5}} = 0.924$$

MNW model

$(\beta^{ij} = 2.1, \zeta^{ij} = 0)$;

$$P_l^{ij} = \frac{120^{-2.1}}{120^{-2.1} + 125^{-2.1}} = \frac{1}{1 + (125/120)^{-2.1}} = 0.521$$

$(\beta^{ij} = 3.7, \zeta^{ij} = 0)$;

$$P_l^{ij} = \frac{120^{-3.7}}{120^{-3.7} + 125^{-3.7}} = \frac{1}{1 + (125/120)^{-3.7}} = 0.538$$

$(\beta^{ij} = 2.1, \zeta^{ij} = 2.5)$;

$$P_l^{ij} = \frac{117.5^{-2.1}}{117.5^{-2.1} + 122.5^{-2.1}} = \frac{1}{1 + (122.5/117.5)^{-2.1}} = 0.523$$

b) Long network

Fig. 2. Two-route networks

To illustrate how the MNW resolves the identical perception variance issue inherited in the MNL model, a two-route network configuration shown in Fig. 2 is adopted. For both networks, the upper route travel time is larger than the lower route travel time by 5 units. However, the upper route travel time is two times larger than the lower route travel time in the short network, while it is only less than 5% larger in the long network. As expected, the MNL model produces the same flow patterns for both short and long networks. This is because the MNL model cannot handle the perception variance with respect to (w.r.t.) different trip lengths. Each route is assumed to have the same (or identical) perception variance of $\pi^2/6\theta^2$, where θ is the logit dispersion parameter, as shown in the upper two panels of Fig. 3. Hence, the MNL probability is solely based on the cost difference irrespective of the overall trip lengths (Sheffi, 1985).

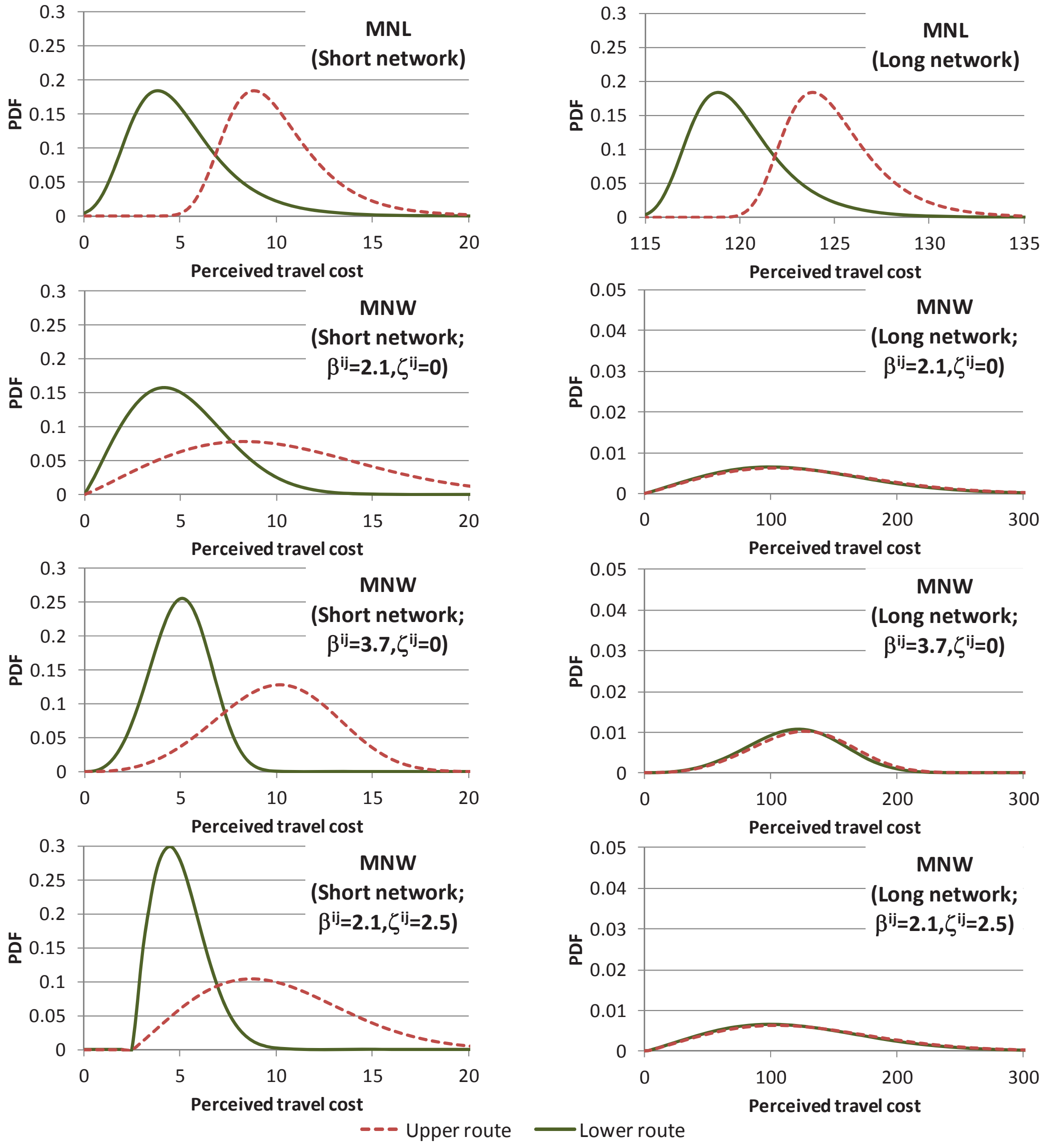


Fig. 3. Perceived travel time distributions for the two-route networks

The MNW model, in contrast, produces different route choice probabilities for the two networks. It uses the *relative* cost difference to differentiate the overall trip length. When considering the perception variance, the MNW model handles the *route-specific* perception variance as a function of β^{ij}, ζ^{ij}, and route travel cost, i.e.,

$$\left(\sigma_r^{ij}\right)^2 = \left[\frac{\left(g_r^{ij} - \zeta^{ij}\right)}{\Gamma\left(1 + 1/\beta^{ij}\right)}\right]^2 \left[\Gamma\left(1 + \frac{2}{\beta^{ij}}\right) - \Gamma^2\left(1 + \frac{1}{\beta^{ij}}\right)\right], \quad \forall r \in R_{ij}, ij \in IJ, \tag{4}$$

where $\Gamma(\ )$ is the gamma function. From Eq. (4), a longer route will have a higher perception variance as shown in Fig. 3. A larger β^{ij} and/or ζ^{ij} will decrease the route perception variance as shown by the PDFs with different combinations of β^{ij} and ζ^{ij}. This will lead to a smaller perception variance among travelers and more flows loaded on the lower-cost route, especially on the network with a shorter overall trip length.

For extreme cases, when $\beta^{ij} \to \infty$ or $\zeta^{ij} \to g_r^{ij}; \forall r \in R_{ij}, ij \in IJ$, the MNW model collapses to the deterministic shortest path problem, where only the lowest-cost route is selected, i.e.,

$$\lim_{\beta^{ij} \to \infty} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = 1; \quad \lim_{\zeta^{ij} \to g_r^{ij}} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = 1, \quad g_r^{ij} < g_l^{ij}, \forall l \neq r \in R_{ij}, ij \in IJ. \tag{5}$$

Meanwhile, as $\beta^{ij} \to 0$, the MNW model collapses to the uniform traffic loading, i.e.,

$$\lim_{\beta^{ij} \to 0} \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} = \frac{1}{\left|R_{ij}\right|}, \quad \forall r \in R_{ij}, ij \in IJ. \tag{6}$$

where $\left|R_{ij}\right|$ is the number of routes connecting O-D pair *ij*.

2.2 PSW model

To relax the *independently distributed* assumption imposed on the MNW model, a path-size factor ϖ_r^{ij} is introduced to the MNW RUM in Eq. (1) to alleviate the route overlapping problem. This path-size factor $\varpi_r^{ij} \in (0,1]$ accounts for different route sizes determined by the length of links within a route and the relative lengths of routes that share a link, i.e., (Ben-Akiva and Bierlaire, 1999)

$$\varpi_r^{ij} = \sum_{a \in \Upsilon_r} \frac{l_a}{L_r^{ij}} \frac{1}{\sum_{k \in R_{ij}} \delta_{ak}^{ij}}, \quad \forall r \in R_{ij}, ij \in IJ, \tag{7}$$

where l_a is the length of link *a*, L_r^{ij} is the length of route *r* connecting O-D pair ij, Υ_r is the set of all links in route *r* between O-D pair *ij*, and δ_{ar}^{ij} is equal to 1 for link *a* on route *r* between O-D pair *ij* and 0 otherwise. The lengths in the common part and the route ratio (i.e., l_a / L_r^{ij}) is a plausibly approximation of the route correlation, and $\sum_{k \in R_{ij}} \delta_{ak}^{ij}$ measures the contribution of link *a* in the route correlation (Frejinger and Bierlaire, 2007). Routes with a heavy overlap with other routes will have a smaller value of ϖ_r^{ij}. The path-size factor can be used to modify the deterministic term of the MNW RUM model in Eq. (1) as follows

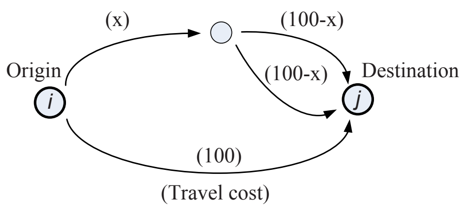
$$U_r^{ij} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}}}{\varpi_r^{ij}} \varepsilon_r^{ij}, \quad \forall r \in R_{ij}, ij \in IJ, \tag{8}$$

which gives the following route choice probability:

$$P_r^{ij} = \frac{\varpi_r^{ij} \left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}, \quad \forall r \in R_{ij}, ij \in IJ. \tag{9}$$

Note that if route r is made up of links belonging exclusively to the route, ϖ_r^{ij} is equal to one, and the PSW model collapses to the MNW model.

To illustrate how the path-size factors handle the route overlapping problem, we use the loop-hole network shown in Fig. 4. In this network, all three routes have equal travel cost. The two upper routes overlap by a portion x, while the lower route is distinct from the two upper routes. According to the *independently distributed* assumption, both MNL and MNW models give the same route choice probability for all x values as shown in Fig. 3. In contrast, the PSW model as well as the path-size logit (PSL) model can handle the route overlapping problem via the path-size factor. As x increases, the probability of choosing the lower route increases. When $x=100$ (i.e., only two routes with equal trip length exist), both upper and lower routes receive the same probability of being selected. Note that the PSW model produces the same results as the PSL model, because all routes have the same travel cost.



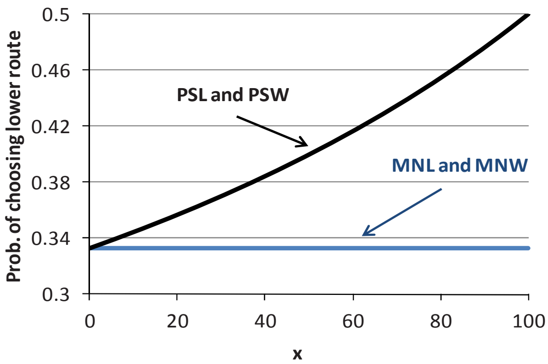


Fig. 4. Loop-hole network

3. Equivalent mathematical programming formulations

This section presents equivalent mathematical programming (MP) formulations for the weibit route choice models under congested networks. A *multiplicative* Beckmann's transformation (MBec) combined with an entropy term are used to develop the MP formulation for the weibit SUE models. Specifically, we present the MP formulations with some qualitative properties. Before presenting the formulations, we describe the necessary assumptions, followed by the MNW-SUE and PSW-SUE models.

3.1 Assumptions

Before formulating the MP formulation for the weibit SUE model, some assumptions are made. To begin with, a general assumption of link travel cost function is made, i.e.,

Assumption 1. The link travel cost τ_a, which could be a function of travel time, is a strictly increasing function w.r.t. its own flow.

Since ζ^{ij} cannot easily be decomposed into the link level, we make another assumption:

Assumption 2. ζ^{ij} is equal to zero.

This assumption indicates that each route is assumed to have the *same* coefficient of variation. From Eq. (4), the route-specific coefficient of variation can be expressed as

$$\vartheta_r^{ij} = \frac{\sigma_r^{ij}}{g_r^{ij}} = \frac{\left(g_r^{ij} - \zeta^{ij}\right)}{g_r^{ij}} \sqrt{\frac{\Gamma\left(1 + 2/\beta^{ij}\right)}{\left(\Gamma\left(1 + 1/\beta^{ij}\right)\right)^2} - 1}, \quad \forall r \in R_{ij}, ij \in IJ. \tag{10}$$

With $\zeta^{ij} = 0$, ϑ_r^{ij} of each route is equal. Note that we can adopt the variational inequality (VI) formulation (e.g., Zhou *et al*., 2008) to incorporate ζ^{ij} in the MNW-SUE model.

Since the weibit model falls within the category of *multiplicative* random utility maximization model (MRUM), the deterministic part of the disutility function is simply a set of *multiplicative* explanatory variables (e.g., Cooper and Nakanishi, 1988). Then, we make an assumption of the route travel cost:

Assumption 3. The route travel cost is a function of multiplicative link travel costs, i.e.,

$$g_r^{ij} = \prod_{a \in \Upsilon_r} \tau_a , \quad \forall r \in R_{ij}, \forall ij \in IJ . \tag{11}$$

This assumption not only maintains the weibit relative cost criterion (Fosgerau and Bierlaire, 2009), but also corresponds to the *Markov process* in transportation network analysis (see Akamatsu, 1996). With a *suitable* multiplicative link cost function, travelers are assumed to make a decision at each node (or *state*) until they reach the destination (or final state) according to the weibit choice probability (see **Appendix B** for details). In other words, travelers are assumed to use the weibit probability to account for the route in further states at each node (current state) for their further choice process (transition of the state).

Following the path-size logit (PSL) SUE formulation provided by Chen *et al.* (2012), the lengths used in the path-size factor for the MP formulation are assumed to be flow independent as follows.

Assumption 4. The lengths l_a and L_r^{ij} used in ϖ_r^{ij} are flow independent.

Note that we can also adopt the VI formulation to incorporate the flow dependent path-size factors (e.g., Zhou *et al*., 2012)

3.2 MNW SUE model

Consider the following MP formulation:

$$\min Z = Z_1 + Z_2$$
$$= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ar}^{ij}} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} \left(\ln f_r^{ij} - 1 \right) \tag{12}$$

s.t.

$$\sum_{r \in R_{ij}} f_r^{ij} = q_{ij} , \quad \forall ij \in IJ , \tag{13}$$

$$f_r^{ij} \geq 0 , \quad \forall r \in R_{ij}, ij \in IJ , \tag{14}$$

where f_r^{ij} is the flow on route *r* between O-D pair *ij*, q_{ij} is the demand between O-D pair *ij*, and v_a is the flow on link *a*. Eq. (13) and Eq. (14) are respectively the flow conservation constraint and the non-negativity constraint. The main differences between this MNW-SUE model and Fisk (1980)'s MNL-SUE model are the *multiplicative* Beckmann's transformation (MBec) in Z_1 and β^{ij} (the perception variance of O-D pair *ij*) in Z_2. The MBec can be converted to an additive form via a log transformation to facilitate the route cost computations, while the O-D specific dispersion parameters β^{ij} is related to the route-specific perception variance (see Eq. (4)). These two differences are the key to the development of the MNW-SUE model. Note that if β^{ij} approaches infinity for all O-D pair *ij*, Z_2 approaches zero. From **Assumption 1**, the log transformation would not alter the results of the MP formulation. Minimizing Z_1 would result in the deterministic user equilibrium (DUE) model where only the lowest-cost routes are used.

Proposition 1. *The MP formulation given in Eqs. (12) through (14) has the solution of the MNW model.*

Proof. Note that the logarithmic term in Eq. (12) implicitly requires τ_a and f_r^{ij} to be positive. By constructing the Lagrangian function of the MNW SUE model and then setting its partial derivative to zero, we obtain

$$\sum_{a\in A}\ln\tau_a\delta_{ra}^{ij}+\frac{1}{\beta^{ij}}\ln f_r^{ij}-\lambda_{ij}=0\,, \tag{15}$$

$$\sum_{r\in R_{ij}} f_r^{ij}=q_{ij}\,, \tag{16}$$

where λ_{ij} is the dual variable associated with the flow conservation constraint in Eq. (13). Eq. (15) can be rearranged as

$$\beta^{ij}\ln\prod_{a\in\Upsilon_r}\tau_a+\ln f_r^{ij}=\beta^{ij}\lambda_{ij}. \tag{17}$$

From **Assumption 3**, Eq. (17) can be expressed as

$$\beta^{ij}\ln g_r^{ij}+\ln f_r^{ij}=\beta^{ij}\lambda\,, \tag{18}$$

which indicates that the MBec provides the *logarithmic* route cost in the equivalency conditions. With this logarithmic route travel cost structure, we have the route flow as a function of $\left(g_r^{ij}\right)^{-\beta^{ij}}$, i.e.,

$$f_r^{ij}=\exp\left(\beta^{ij}\lambda_{ij}\right)\left(g_r^{ij}\right)^{-\beta^{ij}}. \tag{19}$$

From Eq. (16) and Eq. (19), the O-D demand can be written as

$$q_{ij}=\sum_{r\in R_{ij}} f_r^{ij}=\exp\left(\beta^{ij}\lambda_{ij}\right)\sum_{r\in R_{ij}}\left(g_r^{ij}\right)^{-\beta^{ij}}. \tag{20}$$

Dividing Eq. (19) by Eq. (20) leads to the MNW probability expression:

$$P_r^{ij}=\frac{\left(g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k\in R_{ij}}\left(g_k^{ij}\right)^{-\beta^{ij}}}\,,\ \forall r\in R_{ij}, ij\in IJ\,. \tag{21}$$

Thus, the MP formulation given in Eqs. (12) through (14) corresponds to the SUE model for which the route-flow solution is obtained according to the MNW model. This completes the proof. □

From **Proposition 1**, we can see that the first term (i.e., multiplicative Beckmann's transformation: MBec) in Eq. (12) uses the logarithm to handle the *relative difference* mechanism in the MNW model. This mechanism can be viewed from the relation between the MNW-SUE and MNL-SUE models shown in Fig. 5. The MBec is rooted from the relation between the Gumbel and Weibull distributions by applying a log transformation to the Beckmann's transformation (Bec) (Beckmann *et al*., 1956) and incorporating the exponential proportions given by the entropy term in the equivalent conditions to obtain the MNW probability. In other words, by applying a *log transformation* to the MNL travel time, we obtain the MNW model (Castillo *et al*., 2008; Fosgerau and Bierlaire, 2009). This is because the Gumbel distribution can be considered as the log-Weibull distribution (White, 1969).

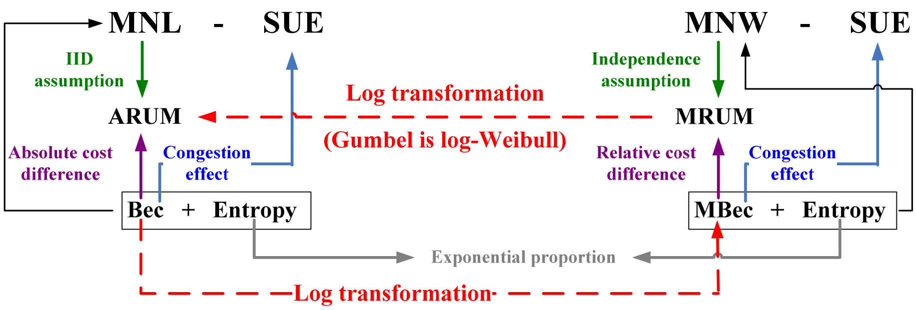


Fig. 5. Relation between MNL-SUE and MNW-SUE models

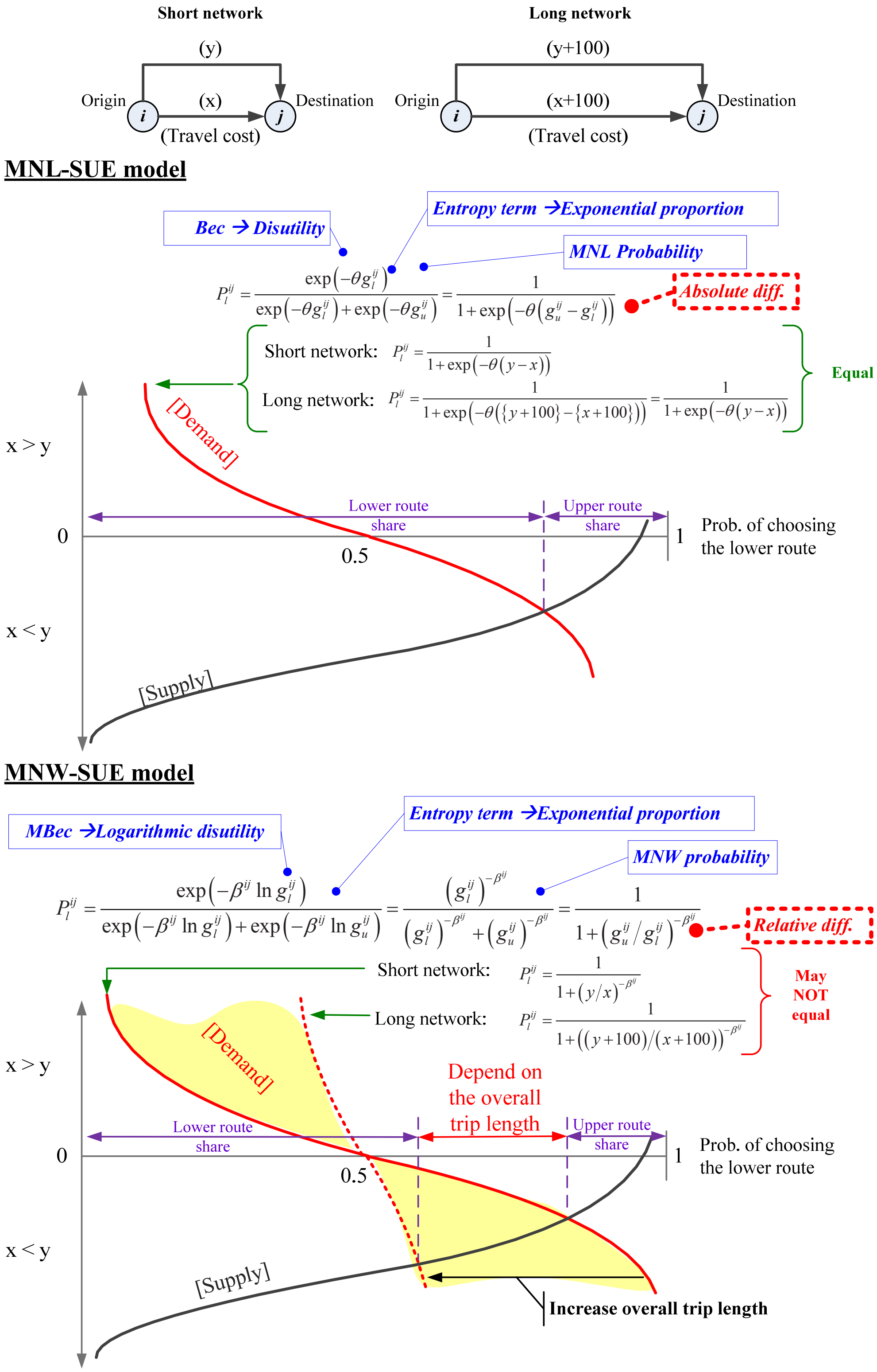


Fig. 6. Visual illustration of the MNL-SUE and MNW-SUE models

To further illustrate the role of MBec in developing the objective function for the MNW-SUE model, we adopt a visual approach used in Bell and Iida (1997) to graphically illustrate the relation between the MNL-SUE and MNW-SUE models in Fig. 6. The supply curve gives the relationship between route flows of the upper and lower routes and their route costs. In the case of monotonically increasing link costs from **Assumption 1**, the supply curve is smooth and exhibits a logistic shape. The demand curve for this two-route network, in relation to the route cost difference between the two routes, is also smooth and logistic, but in opposite direction to the supply curve. The logarithmic route cost produced by the MBec and the exponential proportions given by the entropy term in the equivalent conditions give the MNW solution. With this, the demand curve changes according to the overall trip length. A longer overall trip length has a steeper demand curve, while a shorter overall trip length has a flatter demand curve. Thus, the probabilities of the upper and lower routes become more similar as the overall trip length increases at the equilibrium point where the demand curve intersects the supply curve (shown by the dotted red line of the MNW SUE model in Fig. 6).

Proposition 2. *The solution of MNW-SUE model is unique.*
Proof. It is sufficient to prove that the objective function in Eq. (12) is strictly convex in the vicinity of route flow and that the feasible region is convex. The convexity of the feasible region is assured by the linear equality constraints in Eq. (13). The nonnegative constraint in Eq. (14) does not alter this characteristic.

Hence, the focus is on the properties of the objective function. This is done by proving that the Hessian matrix is positive definite. According to **Assumption 1**, the Hessian matrix of the multiplicative Beckmann's transformation Z_1 is positive semi-definite w.r.t. the route flow variables. This is similar to the Beckmann's transformation case. The Hessian matrix of Z_2 can be shown as

$$\frac{\partial^2 (Z_2)}{\partial f_r^{ij} \partial f_k^{ij}} = \begin{cases} \dfrac{1}{\beta^{ij} f_r^{ij}} > 0 \ ; \ r = k \\ 0 \qquad ; \ otherwise \end{cases} . \tag{22}$$

As such, the Hessian matrix of Z_2 is positive definite. Hence, $Z_1 + Z_2$ is strictly convex. The solution of MNW-SUE model is unique w.r.t. route flows. This completes the proof. □

3.3 PSW SUE model

In this section, we provide an equivalent MP formulation for the PSW-SUE model to consider both route overlapping and heterogeneous perception variance under congested networks. Following the path-size logit (PSL) SUE formulation provided by Chen *et al.* (2012), the PSW-SUE can be formulated as follows:

$$\begin{aligned} &\min Z = Z_1 + Z_2 + Z_3 \\ &= \sum_{a \in A} \int_0^{v_a = \sum_{ij \in IJ} \sum_{r \in R_{ij}} f_r^{ij} \delta_{ra}^{ij}} \ln \tau_a(\omega) d\omega + \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} \left(\ln f_r^{ij} - 1\right) - \sum_{ij \in IJ} \frac{1}{\beta^{ij}} \sum_{r \in R_{ij}} f_r^{ij} \ln \varpi_r^{ij} \end{aligned} \tag{23}$$

subject to the flow conservation and non-negativity constraints in Eq. (13) and Eq. (14). The term Z_3 is introduced to the MNW-SUE formulation in Eq. (12) to capture the length/size of the routes in order to correct the MNW choice probability. Note that when there is no route overlap (i.e., $\varpi_r^{ij} = 1$), the PSW-SUE model collapses to the MNW-SUE model.

Proposition 3. *The MP formulation given in Eqs. (23), (13), and (14) has the solution of the PSW model.*
Proof. Following the same principle as **Proposition 1**, we have

$$\beta^{ij} \ln g_r^{ij} + \ln f_r^{ij} - \ln \varpi_r^{ij} = \beta^{ij} \lambda_{ij}, \tag{24}$$

which gives

$$f_r^{ij} = \exp\left(\beta^{ij} \lambda_{ij}\right) \varpi_r^{ij} \left(g_r^{ij}\right)^{-\beta^{ij}}, \tag{25}$$

$$q_{ij} = \sum_{r \in R_{ij}} f_r^{ij} = \exp\left(\beta^{ij} \lambda_{ij}\right) \sum_{r \in R_{ij}} \varpi_r^{ij} \left(g_r^{ij}\right)^{-\beta^{ij}} . \tag{26}$$

Then, dividing Eq. (25) by Eq. (26) leads to the PSW probability expression:

$$P_r^{ij} = \frac{\varpi_r^{ij} \left(g_r^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \varpi_k^{ij} \left(g_k^{ij}\right)^{-\beta^{ij}}} . \tag{27}$$

Thus, the MP formulation given in Eqs. (23), (13), and (14) corresponds to the SUE model for which the route-flow solution is obtained according to the PSW model. This completes the proof. □

Proposition 4. *The solution of PSW-SUE model is unique.*

Proof. Following the same principle as **Proposition 2**, the Hessian matrices of Z_1 and Z_2 are positive semi-definite and positive definite, respectively. Since ϖ_r^{ij} in Z_3 is flow independent from **Assumption 4**, we have

$$\frac{\partial^2 \left(Z_3\right)}{\partial f_r^{ij} \partial f_k^{ij}} = 0 . \tag{28}$$

Thus, $Z_1 + Z_2 + Z_3$ is strictly convex. The solution of PSW SUE model is unique w.r.t. route flows. This completes the proof. □

4. Solution algorithm

In this study, a path-based algorithm based on the partial linearization method is adopted to solve the PSW-SUE model as shown in Fig. 7. This descent algorithm iterates between the search direction and line search until the stopping criterion of a convex optimization problem is reached (Patriksson, 1994). In the PSW-SUE formulation, the search direction is obtained by solving the first-order approximation of the MBec. For the line search, we consider the classical generalized Armijo rule (Bertsekas, 1976) to find an approximate stepsize, which has been found to be effective in solving the CNL-SUE model (Bekhor *et al.*, 2008). A column generation procedure (Dantzig, 1963) is adopted to resolve the route enumeration issue. Alternatively, a pre-generated working route set based on a behavioral route choice generation method (Prashker and Bekhor, 2004; Prato, 2009) could also be used instead of the column generation procedure.

Since the link cost is in a multiplicative form, we use the logarithm operator to transform it to an additive form, i.e.,

$$\overline{\tau}_a = \ln \tau_a, \ \forall a \in A . \tag{29}$$

This process facilitates the use of an ordinary shortest path algorithm. The route cost can then be determined from

$$g_r^{ij} = \exp\left(\sum_{a \in A} \overline{\tau}_a \delta_{ar}^{ij}\right), \ \forall r \in R_{ij}, ij \in IJ . \tag{30}$$

It should be noted that $\overline{\tau}_a$ could be negative if τ_a is less than one. In such cases, a shortest path algorithm that can handle negative cycles is necessary to avoid an infinite loop. We consider Pape's algorithm (1974), which is a label correcting method that can work with negative link costs. When the network contains a negative cost loop, all negative $\overline{\tau}_a$ would be set to a very small positive number, and the algorithm is then repeated. A more appropriate modification to Pape's algorithm could be implemented to generate the shortest "simple routes" with the presence of negative cycles (e.g., the labeling and scanning method described in Tarjan (1983)). The basic idea is to include a scan operation to the shortest path algorithm to eliminate negative cost cycles. For more information, see Tarjan (1983).

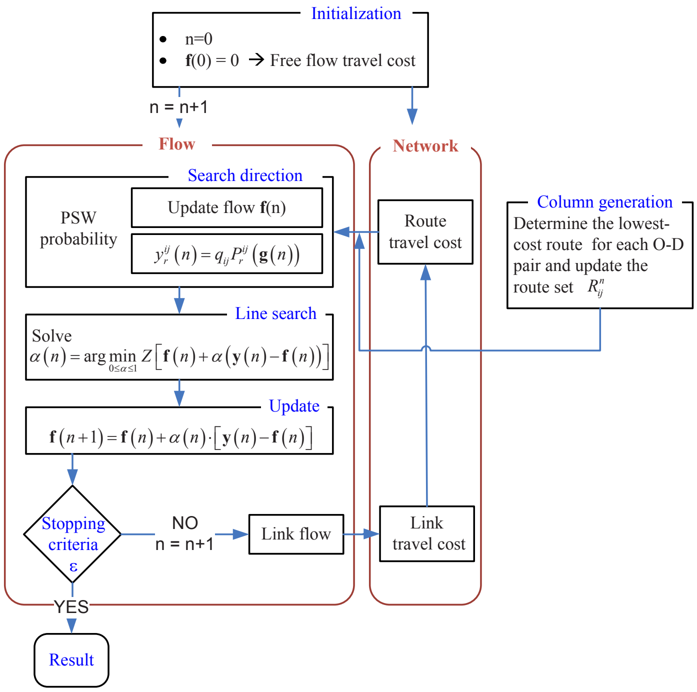


Fig. 7. Partial linearization method for solving the MNW-SUE and PSW-SUE models

5. Numerical results

In this section, we present three numerical examples. Example 1 uses the two-route networks in Fig. 2 with flow dependent travel cost. This example is adopted to investigate the solution from the MNW-SUE model and compare it with the MNL-SUE models (with and without scaling technique). Example 2 is the modified loop-hole network used to consider both route overlapping and route-specific perception variance problems simultaneously. Example 3 is the Winnipeg network used as a case study to demonstrate its applicability in real networks. Without loss of generality, all routes are assumed to have the same coefficient of variation ϑ_r^{ij} of 0.3 (i.e., $\beta^{ij} = 3.7$ for all O-D pairs, see Eq. (10)) unless specified otherwise. l_a and L_r^{ij} used in the path-size factor ϖ_r^{ij} are set to the link free-flow travel cost and route free-flow travel cost, respectively. The dispersion parameter θ of the MNL-SUE model is set to 0.1, and θ of the MNL-SUE model with scaling technique (or MNLs-SUE model) is set corresponding to $\vartheta_r^{ij} = 0.3$ using the uncongested lowest-cost route (Chen *et al.,* 2012). For the MNP-SUE model, the solution is computed by the Monte Carlo stochastic loading technique with 2000 draws to obtain stable results.

5.1 Example 1: Two-route network

The two-route networks in Fig. 2 are modified to incorporate the congested effect as shown in Table 1. The previous route travel cost configuration is used as the free flow travel cost, and $f_r^{ij}/10$ is introduced to create the flow-dependent travel cost. The O-D demand is 100 vehicles per unit time. We first investigate the MNW solution from the MNW-SUE MP formulation, followed by the effect of different trip lengths.

Table 1: Flow-dependent route travel cost for the two-route networks

Network	Upper route	Lower route
Short	$10 + f_u^{ij}/10$	$5 + f_l^{ij}/10$
Long	$125 + f_u^{ij}/10$	$120 + f_l^{ij}/10$

5.1.1 MNW solution

The MNW-SUE objective value for the short network can be expressed as (see Eq. (12))

$$\min \int_0^{f_u^{ij}} \ln\left(10+\frac{\omega}{10}\right)d\omega + \int_0^{f_l^{ij}} \ln\left(5+\frac{\omega}{10}\right)d\omega + \frac{1}{3.7}\left[f_u^{ij}\left(\ln f_u^{ij}-1\right)+f_l^{ij}\left(\ln f_l^{ij}-1\right)\right] \tag{31}$$

s.t.

$$f_u^{ij}+f_l^{ij}=100\,. \tag{32}$$

From Eq. (32), derivative of Eq. (31) w.r.t. the flow on the lower route f_l gives

$$-\ln\left(10+\frac{100-f_l^{ij}}{10}\right)+\ln\left(5+\frac{f_l^{ij}}{10}\right)+\frac{1}{3.7}\left[-\ln\left(100-f_l^{ij}\right)+\ln f_l^{ij}\right]=0\,. \tag{33}$$

Rearranging Eq. (33) gives

$$\frac{f_l^{ij}}{100-f_l^{ij}}=\frac{\left(5+f_l^{ij}/10\right)^{-3.7}}{\left(10+\left(100-f_l^{ij}\right)/10\right)^{-3.7}}\,. \tag{34}$$

Since $10+\left(100-f_l^{ij}\right)/10$ and $5+f_u^{ij}/10$ are respectively the costs of upper and lower routes, Eq. (34) gives the MNW choice probability, i.e.,

$$\frac{f_l^{ij}}{f_u^{ij}}=\frac{\left(g_l^{ij}\right)^{-3.7}}{\left(g_u^{ij}\right)^{-3.7}} \; or \; P_l^{ij}=\frac{f_l^{ij}}{f_l^{ij}+f_u^{ij}}=\frac{\left(g_l^{ij}\right)^{-3.7}}{\left(g_l^{ij}\right)^{-3.7}+\left(g_u^{ij}\right)^{-3.7}}\,. \tag{35}$$

This result indicates that the multiplicative Beckmann's transformation preserves the relative cost difference criterion of the weibit model. By solving Eq. (34), we obtain the route flow solution of the MNW-SUE problem, i.e.,

$$f_u^{ij}=35.25;\; f_l^{ij}=64.75\,. \tag{36}$$

5.1.2 Effect of different trip lengths

Next, we consider the effect of different trip lengths under the SUE framework. The results are shown in Table 2. As expected, the MNL-SUE model produces the same flow pattern for both short and long networks, regardless of the overall trip length. Meanwhile, the MNLs-SUE and MNW-SUE models assign different flow patterns to reflect the overall trip lengths. Specifically, both models assign a smaller amount of traffic flows on the lower route as the overall trip length increases. These assignment results are consistent with that there is higher opportunity for "*wrong*" perception (in the sense of choosing a larger-cost route) on the longer overall trip length network (Sheffi, 1985). Note that the MNLs-SUE model has a higher amount of flows on the lower route than the MNW-SUE model since the MNLs-SUE model still retains the identically distributed assumption; each route has the same and fixed perception variance from the classical logit assumption, despite different scaling factors (i.e., $\theta = 0.86$ for the short network and $\theta = 0.04$ for the long network) are used in the two networks.

Table 2: Results of the two-route networks

	Model	MNL-SUE	MNLs-SUE	MNW-SUE
Short network	Flow on the upper route	*41.72*	29.96	35.25
	Flow on the lower route	*58.28*	70.04	64.75
Long network	Flow on the upper route	*41.72*	46.23	46.84
	Flow on the lower route	*58.28*	53.77	53.16

The objective values (Z=Z_1+Z_2) of all three models are presented in Table 3. All models have a higher total objective value (Z) as the overall trip length increases. The Bec value (Z_1) of the MNL-SUE and MNLs-SUE models are higher than the MBec value (Z_1) of the MNW-SUE model. This is because the MBec has the logarithm transformation. Note that the Z_1/Z_2 ratio of the MNLs-SUE model is different from the Z_1/Z_2

ratio of the MNW-SUE model. While the Z_1/Z_2 ratio of the MNLs-SUE model is smaller as the overall trip length increases, the Z_1/Z_2 ratio of the MNW-SUE model is larger as the overall trip length increases. These results appear to indicate that the MNW-SUE model uses the Z_1/Z_2 ratio differently to capture the effect of different trip lengths compared to that of the MNL-SUE model with scaling.

Table 3: Objective values of MNL-SUE, MNLs-SUE and MNW-SUE models for the two-route networks

Network	OBJ	MNL-SUE	MNLs-SUE	MNW-SUE
Short	Bec or Mbec (Z_1)	965.45	939.96	382.27
	Entropy term (Z_2)	3925.80	467.19	106.92
	Total OBJ value ($Z=Z_1+Z_2$)	4891.25	1407.15	489.20
	Bec or Mbec and Entropy term (Z_1/Z_2)	0.25	*2.01*	*3.58*
Long	Bec or Mbec (Z_1)	12465.45	12482.55	9813.07
	Entropy term (Z_2)	3925.80	10988.69	105.78
	Total OBJ value ($Z=Z_1+Z_2$)	16391.25	23471.25	9918.86
	Bec or Mbec and Entropy term (Z_1/Z_2)	3.18	*1.14*	*92.76*

5.2 Example 2: Modified loop-hole network

Then, we use the modified loop-hole network given in Fig. 8 to consider both route overlapping and route-specific perception variance problems and the effect of demand level and coefficient of variation. This network has three routes. The upper routes have a fixed overlapping section by half of the route free flow travel time (FFTT). The lower route is truly independent, and its FFTT can be varied according to $y \in [0,10]$. All links have the same capacity of 100 vehicle per hour (vph), and the O-D demand is 100 vph.

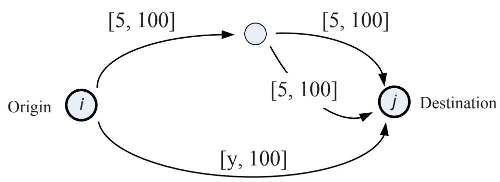


Fig. 8. Modified loop-hole network

The flow dependent link travel time is represented by the standard Bureau of Public Road (BPR) function

$$t_a = t_a^0 \left[1 + 0.15 \left(\frac{v_a}{c_a} \right)^4 \right], \tag{37}$$

where t_a^0 is the FFTT on link a, and c_a is the capacity (in vph) on link a. Without loss of generality, we assume that the link travel cost (or disutility) is an exponential function (Hensher and Truong, 1985; Polak, 1987; Mirchandani and Soroush, 1987), i.e.,

$$\tau_a = e^{0.075 t_a}, \quad \forall a \in A. \tag{38}$$

5.2.1 Effect of overlapping and heterogeneous perception variance

We first consider the route overlapping and route-specific perception variance problems. The results in Fig. 9 show that all SUE models assign a smaller amount of flows on the lower route when y increases. While the MNP-SUE and PSW-SUE models seem to give similar traffic flow patterns, the MNW-SUE model assigns a smaller flow on the lower route. This is because the MNW-SUE model does not handle the route overlapping problem. With the independently distributed assumption, the MNW-SUE model considers each route as an independent alternative. As such, it assigns more flow on the routes with overlapping, hence a smaller amount of flow on the lower route.

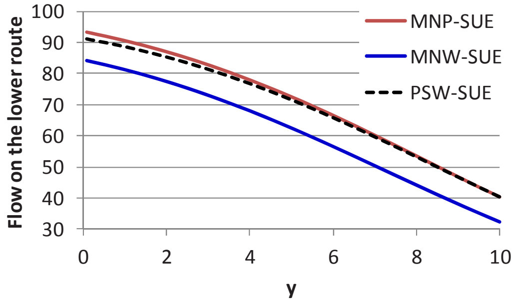


Fig. 9. Traffic flow patterns of the modified loop-hole network

5.2.2 Effect of demand levels and coefficient of variations

We continue to use the modified loop-hole network with $y = 5$ to investigate the effect of demand levels and coefficient of variations. The O-D demand is varied from 25 to 300 vph, and ϑ_r^{ij} is varied from 0.1 to 1. The root mean square error (RMSE) is used as a statistical measure to compare the difference between the PSW-SUE model relative to the user equilibrium (UE) model, i.e.,

$$RMSE = \sqrt{\sum_{a \in A} \frac{\left(v_a^{\{UE\}} - v_a^{\{PSW\}}\right)^2}{|A|}}, \tag{39}$$

where $|A|$ is the number of links in the network (i.e., 4 links). A low value of RMSE means that both assignment models perform similarly.

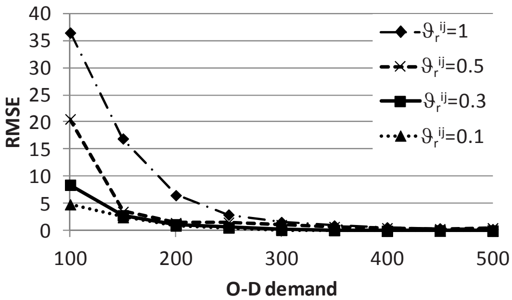


Fig. 10. Effects of demand levels and coefficients of variations ϑ_r^{ij}

It can be seen from Fig. 10 that as the demand level increases, the RMSE decreases. This means that the PSW-SUE model approaches the UE model when the congestion level is increased (i.e., congestion effect due to high demand levels of 400 to 500 vph dominates the solution). Also, the RSME decreases when ϑ_r^{ij} decreases (i.e., β^{ij} increases or lower perception variance). The PSW-SUE flow patterns also tend to the UE flow pattern. This implies the demand is more concentrated on the minimum cost routes (i.e., travelers are able to select the lower-cost routes more often since they have better knowledge of the network traffic conditions). Otherwise, the two models will produce different flow patterns for low demand levels and larger ϑ_r^{ij} values.

5.3 Example 3: Winnipeg network

This example adopts the Winnipeg network (shown in Fig. 11) as a case study to demonstrate its applicability in a real network. This network consists of 154 zones, 2,535 links, and 4,345 O-D pairs. The network topology, link characteristics, and O-D demands can be found in Emme/2 software (INRO Consultants, 1999). We continue to use the link travel cost configuration in Eq. (38) for this real network.

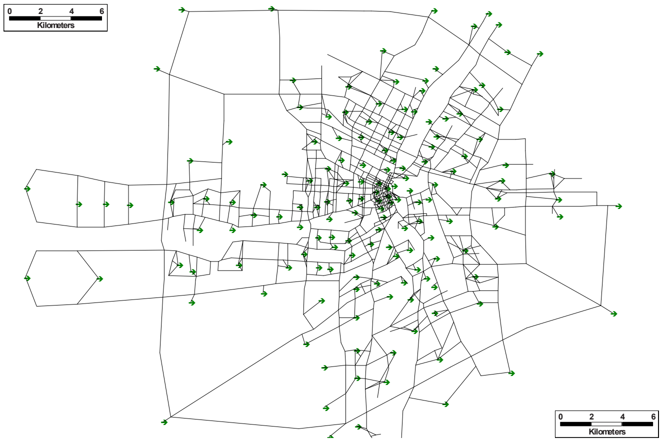


Fig. 11. Winnipeg network

5.3.1 Computational results

For the stopping criterion of the path-based partial linearization algorithm, we adopt the residual error defined as follows.

$$\varepsilon = \frac{\sum_{ij \in IJ} \sum_{r \in R_{ij}} \left(gc_r^{ij} - \min\left(gc_r^{ij} \right) \right) f_r^{ij}}{\sum_{ij \in IJ} \sum_{r \in R_{ij}} gc_r^{ij} f_r^{ij}}, \tag{40}$$

where $gc_r^{ij} = \beta^{ij} \ln g_r^{ij} + \ln f_r^{ij} - \ln \varpi_r^{ij}$, which should be identical for all routes at equilibrium. Without loss of generality, ε is set at 10^{-8}. The convergence characteristics of the path-based partial linearization algorithm are shown in Fig. 12 and Table 4. From Fig. 12, it appears that the partial linearization algorithm can solve both SUE models (MNW and PSW) in a linear convergence rate, with the PSW-SUE model requiring a few more iterations to reach the desired level of accuracy. The computational efforts required by each SUE model are provided in Table 4. As expected, the PSW-SUE model does require slightly more computational efforts than the MNW-SUE model in terms of number of iterations, CPU time, and CPU time per iteration.

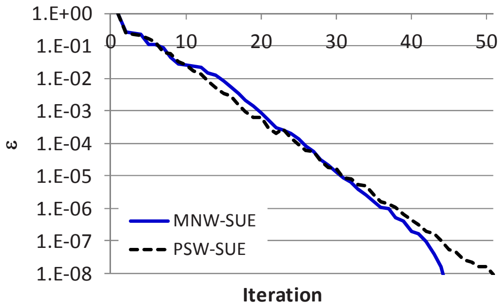


Fig. 12. Convergence characteristics of the path-based partial linearization algorithm

Table 4. Computational efforts of the MNW-SUE and PSW-SUE models

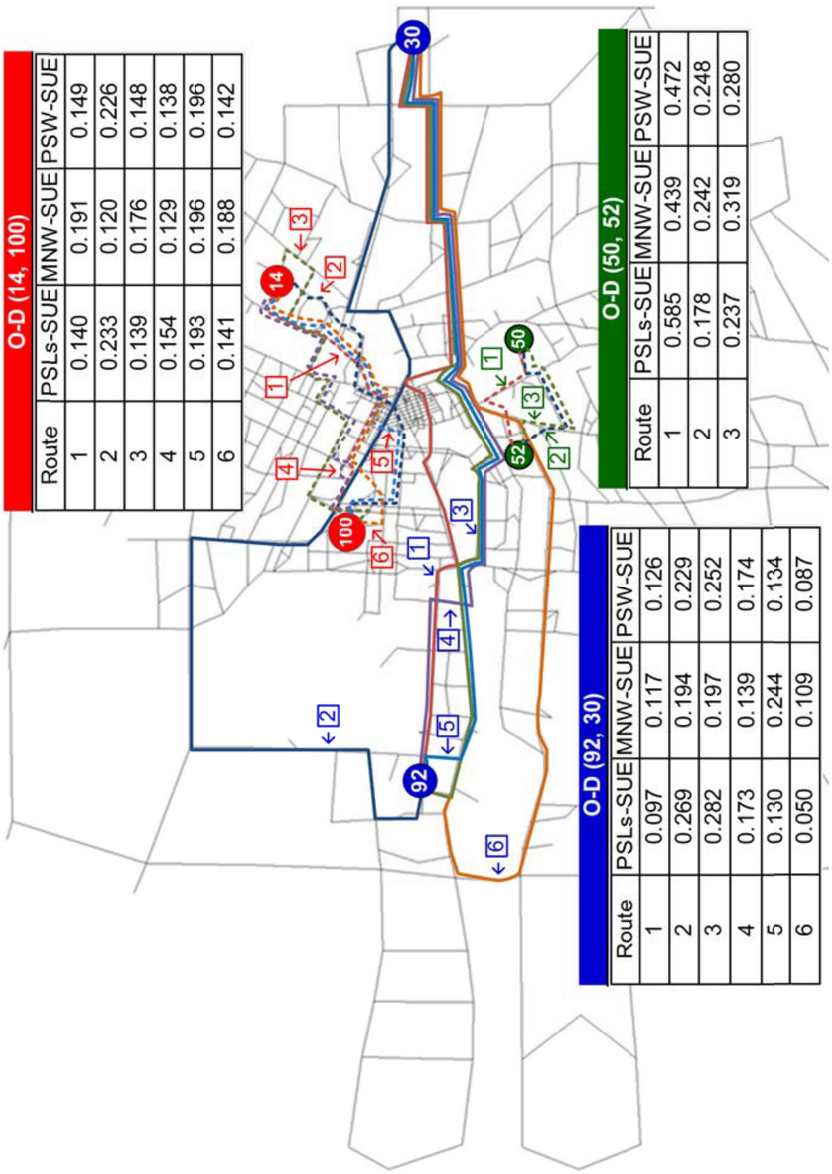
Model	# of iterations	CPU time (sec)*	CPU time per iteration	# of routes
MNW-SUE	45	10.04	0.22	15,791
PSW-SUE	51	13.41	0.26	15,442

* All the algorithms are coded in Compaq Visual FORTRAN 6.6 and run on a personal computer with 3.8 G Pentium-IV processor and 2G RAM

5.3.2 Flow allocation comparison

At the disaggregate level, we examine the route choice probabilities produced by the PSLs-SUE, MNW-SUE and PSW-SUE models. Note that the PSLs-SUE model use the same link travel cost configuration as the weibit SUE model in Eq. (38). For demonstration purposes, we use O-D pairs (50, 52), (14, 100), and (92, 30) to respectively represent a short, medium, and long O-D pair. The route choice probabilities shown in Fig. 13 are under the respective equilibrium route flow pattern. Recall that each SUE model handles the IID assumption (i.e., route overlapping and non-identical perception variance problems) differently as follows: (1) The PSLs-SUE model can handle the route overlapping problem, but it cannot capture the heterogeneous perception variance among different routes, (2) the MNW-SUE model can handle the heterogeneous perception variance among different routes, but it cannot capture the route overlapping issue, while (3) the PSW-SUE model consider both route overlapping and heterogeneous perception variance problems simultaneously. Thus, different route flows (or probabilities) can be expected. Even though O-D pair (50, 52) has only 3 routes with a heavy overlap between route 2 and route 3, the three SUE models produce significantly different results. The PSLs-SUE model assigns a higher probability to the independent route (i.e., route 1); the MNW SUE model, on the other hand, assigns a higher probability to the two overlapping routes (i.e., route 2 and route 3) compared to the PSL-SUE model; and the PSW-SUE model seems to allocate a flow pattern in between these two models by accounting for both overlap and heterogeneous perception variance among different routes. For the two longer O-D pairs ((14, 100) and (92, 30)), more routes are generated as a result of a longer trip length. When both number of routes and trip length are increasing, the differences among the three models also decrease.

At the aggregate level, we examine the effect of route overlapping and heterogeneous perception variance problems on the link flow patterns. The link flow pattern difference between the PSLs-SUE and PSW-SUE models can be found mostly in the central business district (CBD) area as shown in Fig. 14. The absolute maximum flow difference in the CBD area is 482 vph compared to 304 vph in the outer area (or non-CBD area). This is because there are many short O-D pairs in the CBD area with different trip lengths that the PSLs-SUE model has difficulty in handling the heterogeneous perception variance among different routes. When comparing between the MNW-SUE and PSW-SUE models, the link flow difference can also be found mostly in the CBD area in Fig. 15. This is because more than 60% of the routes (or more than 9,000 routes) pass through the CBD area. As a result, route overlapping is a significant problem in the CBD area compared to the outer area.



O-D (14, 100)

Route	PSLs-SUE	MNW-SUE	PSW-SUE
1	0.140	0.191	0.149
2	0.233	0.120	0.226
3	0.139	0.176	0.148
4	0.154	0.129	0.138
5	0.193	0.196	0.196
6	0.141	0.188	0.142

O-D (50, 52)

Route	PSLs-SUE	MNW-SUE	PSW-SUE
1	0.585	0.439	0.472
2	0.178	0.242	0.248
3	0.237	0.319	0.280

O-D (92, 30)

Route	PSLs-SUE	MNW-SUE	PSW-SUE
1	0.097	0.117	0.126
2	0.269	0.194	0.229
3	0.282	0.197	0.252
4	0.173	0.139	0.174
5	0.130	0.244	0.134
6	0.050	0.109	0.087

Fig. 13. Comparison of route choice probabilities of three O-D pairs

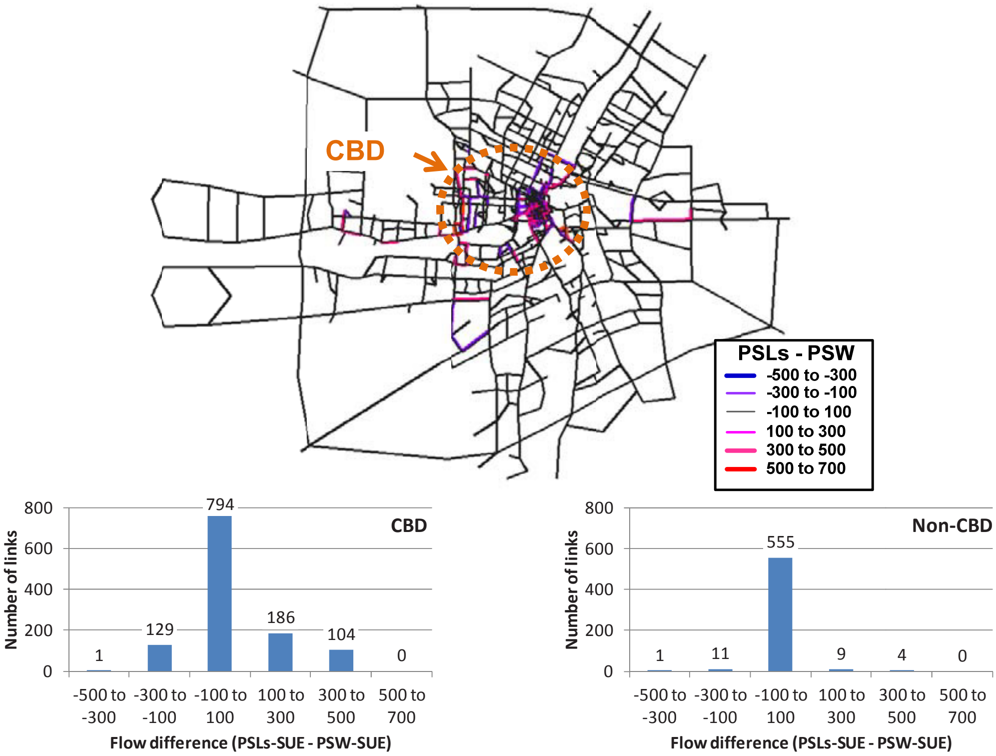


Fig. 14. Link flow difference between PSLs-SUE and PSW-SUE models

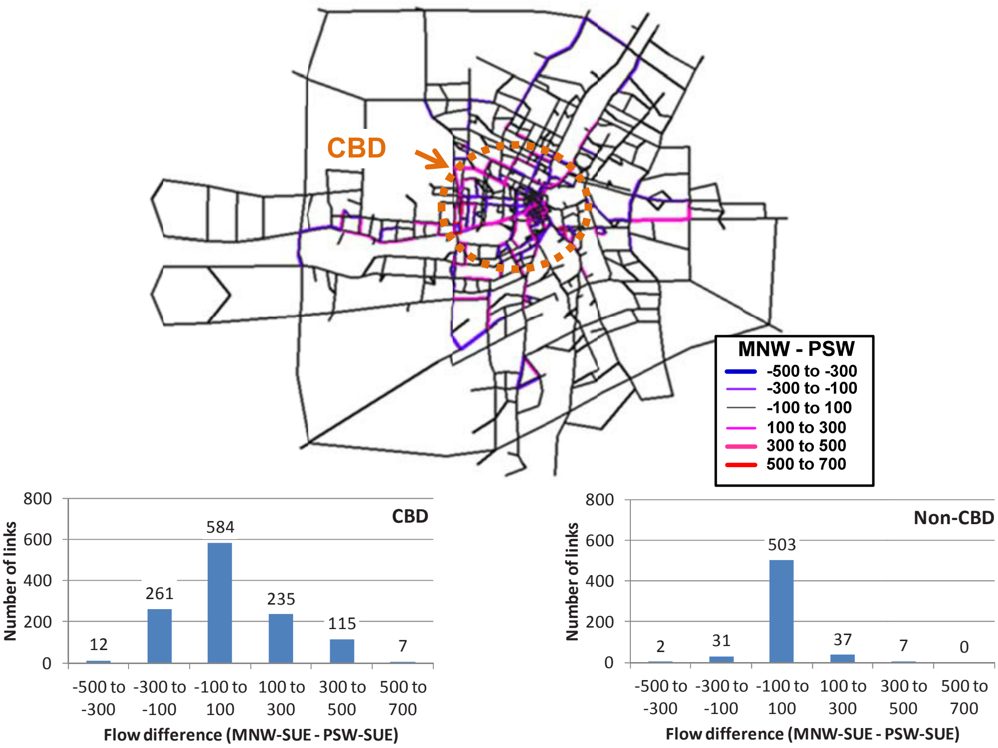


Fig. 15. Link flow difference between MNW-SUE and PSW-SUE models

6. Concluding remarks

In this paper, we presented a path-size weibit (PSW) route choice model with an equivalent mathematical programming stochastic user equilibrium (SUE) formulation to relax the *independently and identically distributed* (IID) assumption imposed on the MNL-SUE model. The proposed route choice model adopts the Weibull distributed random error term to handle the route-specific perception variance as a function of route travel cost and a path-size factor to resolve the route overlapping issue by adjusting the choice probabilities for routes with strong couplings with other routes. A multiplicative Beckmann's transformation (MBec) was developed to handle the multiplicative nature of this new route choice model. Incorporating this MBec with an entropy term gives the PSW traffic flow solution under congested conditions. The PSW-SUE model was tested on three networks to examine its features in comparison with some existing SUE models (MNL, PSL, MNW, and MNP) and its applicability on a real network. Through the numerical results, we observed the followings:

- The MNW-SUE model (without the *identically distributed* assumption) can account for the overall trip length by using the relative cost difference to determine the flow pattern much better than the MNL-SUE model does.
- The PSW-SUE model can produce a compatible traffic flow pattern compared to the MNP-SUE model in a congested network.
- The PSW-SUE model can be applied in a real network as shown by the Winnipeg network.

For future research, parameter calibration should be conducted for the PSW-SUE model, and more tests should be conducted to validate the usefulness of the PSW-SUE model. Moreover, the PSW-SUE model should be extended to consider non-zero location parameter, flow-dependent path-size factors, multiple user classes, and other travel choice dimensions (e.g., elastic demand for travel choice, modal split for mode choice, and trip distribution for destination choice).

Acknowledgements

The authors are grateful to three anonymous referees for their constructive comments and suggestions to improve the quality and clarity of the paper. The first author would like to acknowledge the financial support from the Royal Thai Government Scholarship.

References

Akamatsu, T., 1996. Cyclic flows, Markov process and stochastic traffic assignment. *Transportation Research Part B*, 30(5), 369–386.

Beckmann, M.J., McGuire, C.B. and Winsten, C.B., 1956. Studies in Economics of Transportation. New Haven: Yale University Press, Conn.

Bekhor S. and Prashker J., 2001. A stochastic user equilibrium formulation for the generalized nested logit model. *Transportation Research Record* 1752, 84-90.

Bekhor, S. and Prashker, J.N., 1999. Formulations of extended logit stochastic user equilibrium assignments. Proceedings of the 14th International Symposium on Transportation and Traffic Theory, Jerusalem, Israel, 351-372.

Bekhor, S., Toledo, T. and Reznikova, L., 2008. A path-based algorithm for the cross-nested logit stochastic user equilibrium. *Computer-Aided Civil and Infrastructure Engineering*, 24(1), 15-25.

Bell, M.G.H. and Iida, Y., 1997. *Transportation Network Analysis*. Wiley, Chichester, West Sussex.

Ben-Akiva, M. and Bierlaire, M., 1999. *Discrete choice methods and their applications to short term travel decisions*. Handbook of Transportation Science, R.W. Halled, Kluwer Publishers.

Ben-Akiva, M. and Lerman, S.R., 1985. *Discrete choice analysis*. Cambridge, MA: MIT Press.

Bertsekas, D.R., 1976. On the Goldstein-Levitin-Polyak gradient projection method. *IEEE Transactions on automatic control*, 21(2), 174-184.

Cascetta, E., Nuzzolo, A., Russo, F. and Vitetta, A., 1996. A modified logit route choice model overcoming path overlapping problems: specification and some calibration results for interurban networks. In

Proceedings of the 13th International Symposium on Transportation and Traffic Theory, Leon, France, 697-711.

Cascetta, E., Russo, F., Viola, F.A., Vitetta, A., 2002. A model of route perception in urban road networks. *Transportation Research Part B*, 36(7), 577-592.

Castillo, E., Menéndez, J. M., Jiménez, P. and Rivas, A., 2008. Closed form expression for choice probabilities in the Weibull case, *Transportation Research Part B,* 42(4), 373-380.

Chen, A., Pravinvongvuth, S., Xu, X., Ryu, S. and Chootinan, P., 2012. Examining the scaling effect and overlapping problem in logit-based stochastic user equilibrium models. *Transportation Research Part A,* 46(8), 1343-1358.

Chu, C. 1989. A paired combinatorial logit model for travel demand analysis. In *Proc., Fifth World Conference on Transportation Research,* Ventura, Calif., 4, 295-309.

Cooper, L.G. and Nakanishi, M., 1988. *Market-share analysis: evaluating competitive marketing effectiveness*. Kluwer Academic Publishers, London.

Daganzo, C.F. and Sheffi, Y., 1977. On stochastic models of traffic assignment. *Transportation Science,* 11(3), 253-274.

Dantzig, G.B., 1963. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ.

Dial, R., 1971. A probabilistic multipath traffic assignment model which obviates path enumeration. *Transportation Research,* 5(2), 83-111.

Fisk, C., 1980. Some developments in equilibrium traffic assignment. *Transportation Research Part B,* 14(3), 243-255.

Fosgerau, M. and Bierlaire, M., 2009. Discrete choice models with multiplicative error terms. *Transportation Research Part B,* 43(5), 494-505.

Frejinger, E. and Bierlaire, M., 2007. Capturing correlation with subnetworks in route choice models, *Transportation Research Part B*, 41(3), 363-378.

Genius, M. and Strazzera, E., 2002. A note about model selection and tests for non-tested contingent valuation models. *Economics Letters,* 74, 363–370.

Hensher, D.A. and Truong T.P., 1985. Valuation of Travel Times Savings. *Journal of Transport Economics and Policy*, 237-260.

INRO Consultants, 1999. Emme/2 User's Manual: Release 9.2. Montréal.

Looney, S.W., 1983. The use of the Weibull distribution in bioassay. *Proceedings of the Statistical Computing Section*, American Statistical Association, 272-277.

Maher, M., 1992. SAM – A stochastic assignment model. In: *Mathematics in Transport Planning and Control*, ed. J.D. Griffiths. Oxford University Press, Oxford.

McFadden, D., 1978. Modeling the choice of residential location. In Anders Karlqvist, Lars Lundqvist, Folke Snickars, and Jorgen Weibull, eds., Special interaction theory and planning models. Amsterdam: North-Holland, 75–96.

Mirchandani, P. and Soroush, H., 1987. Generalized traffic equilibrium with probabilistic travel times and perceptions. *Transportation Science*, 21 (3), 133-152.

Pape, U., 1974. Implementation and efficiency of Moore algorithms for the shortest route problem. Mathematical Programming 7, 212-222.

Patriksson, M., 1994. *The Traffic Assignment Problem: Models and Methods*. VSP, Utrecht, The Netherlands .

Polak, J., 1987. A more general model of individual departure time choice. In: PTRC Summer Annual Meeting, Proceedings of Seminar C.

Prashker, J.N. and Bekhor, S., 2004. Route choice models used in the stochastic user equilibrium problem: a review. *Transport Reviews* 24(4), 437-463.

Prato, C.G., 2009. Route choice modeling: past, present and future research directions. *Journal of Choice Modelling* 2(1), 65-100.

Rosa, A., Maher, M.J., 2002. Algorithms for solving the probit path-based stochastic user equilibrium traffic assignment problem with one or more user classes. *Proceedings of the 15th International Symposium on Transportation and Traffic Theory*, Australia, 371-392.

Sheffi, Y. and Powell, W.B., 1982. An algorithm for the equilibrium assignment problem with random link times. *Networks,* 12(2), 191–207.

Sheffi, Y., 1985. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice-Hall.

Strong A.W., Wu, E.Y., Vollertsen, R.P., Sune, J., Rossa, G.L., Sullivan, T.D. and Rauch, S.E. (2009) *Reliability Wearout Mechanisms in Advanced CMOS Technologies*. Wiley.

Tarjan, R.E., 1983. Data Structures and Network Algorithms. CBMS 44. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Vovsha, P., 1997. Application of Cross-Nested Logit Model to Mode Choice in Tel Aviv, Israel, Metropolitan Area. In *Transportation Research Record 1607,* TRB, National Research Council, Washington, D.C., 6–15.

Wen, C., and Koppelman, F., 2000. Generalized Nested Logit Model. Presented at the 79th Annual Meeting of the Transportation Research Board, Washington, D.C.,

White, J.S., 1969. The moments of log-Weibull order statistics. *Technometrics* 11, 373-386.

Zhou, Z., Chen, A. and Behkor, S., 2012. C-logit stochastic user equilibrium model: formulations and solution algorithm. *Transportmetrica,* 8(1), 17-41.

Zhou, Z., Chen, A. and Wong S.C., 2008. Alternative formulations of a combined trip generation, trip distribution, modal split, and traffic assignment model. *European Journal of Operational Research*, 198(1), 129-138.

Appendix A

We can determine the choice probability by (e.g., Ben-Akiva and Lerman, 1985; Castillo *et al*., 2008)

$$P_r^{ij} = -\int_{-\infty}^{+\infty} \bar{H}_r^{ij}\left(..,\varepsilon_r^{ij},..\right) d\varepsilon_r^{ij},\ \forall r \in R_{ij}, ij \in IJ, \tag{41}$$

where $\bar{H}_r^{ij}$ is the partial derivative of the joint survival function w.r.t. ε_r^{ij}. For the weibit RUM model, the CDF of each random error term is

$$F\left(\varepsilon_r^{ij}\right) = 1 - \exp\left(-\varepsilon_r^{ij}\right), \tag{42}$$

and hence the suvival function of each random error term is

$$\bar{F}\left(\varepsilon_r^{ij}\right) = 1 - F\left(\varepsilon_r^{ij}\right) = \exp\left(-\varepsilon_r^{ij}\right). \tag{43}$$

Under the independence assumption, the joint survival function can be expressed as

$$\bar{H} = \prod_{r \in R_{ij}} \bar{F}\left(\varepsilon_r^{ij}\right) = \exp\left(-\sum_{r \in R_{ij}} \varepsilon_r^{ij}\right), \tag{44}$$

which gives

$$\bar{H}_r^{ij} = -\exp\left(-\sum_{r \in R_{ij}} \varepsilon_r^{ij}\right). \tag{45}$$

From Eq. (2), Eq. (45) can be restated as

$$\bar{H}_r^{ij} = -\exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right). \tag{46}$$

Then, substituting Eq. (45) in Eq. (41) gives

$$\begin{aligned} P_r^{ij} &= \int_0^{\infty} \exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right) d\varepsilon_r^{ij} \\ &= -\frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} \left[\exp\left(-\left(g_r^{ij} - \zeta^{ij}\right)^{\beta^{ij}} \varepsilon_r^{ij} \sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}\right)\right]_0^{\infty}. \\ &= \frac{\left(g_r^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}}{\sum_{k \in R_{ij}} \left(g_k^{ij} - \zeta^{ij}\right)^{-\beta^{ij}}} \end{aligned} \tag{47}$$

Appendix B

In this appendix, we show how the weibit route choice decision is related to the *Markov* process (Markov chain) in the network analysis framework according to **Assumption 3**. According to the MNW choice probability, the probability of selecting a succeeding node can be expressed as

$$P_{p\to q}^{ij} = \sum_{\substack{r\in R_{pj} \\ \text{pass on node } p,q}} \frac{\left(g_r^{pj}\right)^{-\beta^{ij}}}{\sum_{k\in R_{pj}} \left(g_k^{pj}\right)^{-\beta^{ij}}}, \tag{48}$$

where i is the origin node, j is the destination node, p denotes the current node in which travelers is about to leave, q denotes the succeeding node in which travelers is about to go, g_r^{pj} is the route travel cost between node p to destination j, and R_{pj} is the set of routes between node p and destination j. Eq. (48) indicates the weibit choice decision. Travelers are assumed to make a decision at each node from available routes following node q ($r\in R_{pj}$ pass on nodes p and q) and available routes following the current node p as shown in Fig. 16.

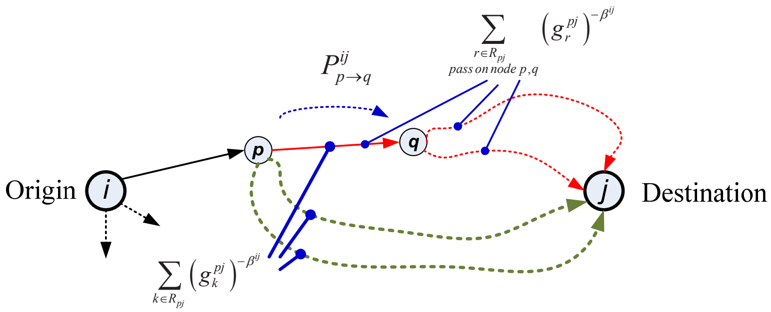


Fig. 16. Conceptual framework for the node-to-node weibit choice behavior

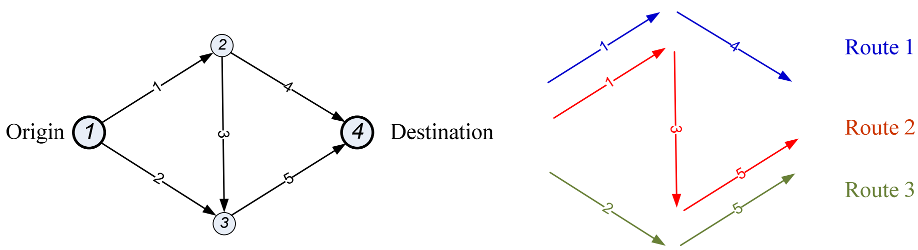


Fig. 17. Braess network

We use the Braess network in Fig. 17 to show that Eq. (48) leads to the MNW route choice probability. The travel cost of each link in this network is τ_a. From **Assumption 3**, the probability of choosing from a node to another node can be expressed as

$$P_{1\to 2}^{14} = \frac{\left(\tau_1\tau_4\right)^{-\beta^{14}} + \left(\tau_1\tau_3\tau_5\right)^{-\beta^{14}}}{\left(\tau_1\tau_4\right)^{-\beta^{14}} + \left(\tau_1\tau_3\tau_5\right)^{-\beta^{14}} + \left(\tau_2\tau_5\right)^{-\beta^{14}}}, \tag{49}$$

$$P_{1\to 3}^{14} = \frac{\left(\tau_2\tau_5\right)^{-\beta^{14}}}{\left(\tau_1\tau_4\right)^{-\beta^{14}} + \left(\tau_1\tau_3\tau_5\right)^{-\beta^{14}} + \left(\tau_2\tau_5\right)^{-\beta^{14}}}, \tag{50}$$

$$P_{2\to 3}^{14} = \frac{\left(\tau_3\tau_5\right)^{-\beta^{14}}}{\left(\tau_4\right)^{-\beta^{14}} + \left(\tau_3\tau_5\right)^{-\beta^{14}}}, \tag{51}$$

$$P_{2\to4}^{14} = \frac{(\tau_4)^{-\beta^{14}}}{(\tau_4)^{-\beta^{14}} + (\tau_3\tau_5)^{-\beta^{14}}}, \tag{52}$$

$$P_{3\to4}^{14} = 1. \tag{53}$$

Assuming that the decision is independent of the preceding decision (on the previous node), the choice probability of each route can be determined from

$$P_r^{ij} = \prod_{pq}\left(P_{p\to q}^{ij}\right)^{\delta_{r,pq}^{ij}}, \tag{54}$$

where $\delta_{r,pq}^{ij}$ equal to 1 if nodes p and q are on the route r between O-D pair ij and 0 otherwise. From Eq. (54), we have the choice probability for route 1 as

$$\begin{aligned} P_1^{14} &= P_{1\to2}^{14} \bullet P_{2\to4}^{14} \\ &= \frac{(\tau_1\tau_4)^{-\beta^{14}} + (\tau_1\tau_3\tau_5)^{-\beta^{14}}}{(\tau_1\tau_4)^{-\beta^{14}} + (\tau_1\tau_3\tau_5)^{-\beta^{14}} + (\tau_2\tau_5)^{-\beta^{14}}} \frac{(\tau_4)^{-\beta^{14}}}{(\tau_4)^{-\beta^{14}} + (\tau_3\tau_5)^{-\beta^{14}}} \\ &= \frac{(\tau_1\tau_4)^{-\beta^{14}}}{(\tau_1\tau_4)^{-\beta^{14}} + (\tau_1\tau_3\tau_5)^{-\beta^{14}} + (\tau_2\tau_5)^{-\beta^{14}}} \\ &= \frac{(g_1^{14})^{-\beta^{14}}}{\sum_{k\in R_{14}} (g_k^{14})^{-\beta^{14}}}, \end{aligned} \tag{55}$$

which corresponds to the MNW probability in Eq. (3). Using the same principle, we have the MNW choice probability of each route. With this, if we assume that each node is a "*state*", and travelers' movement between two adjacent nodes corresponds to the transition of the state according to $P_{p\to q}^{ij}$, we have Akamatsu (1996)'s Makov chain for the transportation network analysis as shown in Fig. 18. Note that, for the PSW case, we need to decompose the path-size factor into the link level before using the principle as presented in this Appendix.

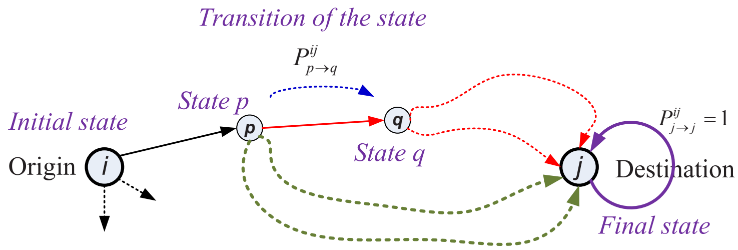


Fig. 18. Markov chain and the node-to-node weibit choice behavior